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Relativistically rotating dust

By Gernot Neugebauer, Andreas Kleinwächter and Reinhard Meinel

Max–Planck–Gesellschaft, Arbeitsgruppe Gravitationstheorie an der Universität Jena,
Max–Wien–Platz 1, D–07743 Jena, Germany

Abstract. Dust configurations play an important role in astrophysics and are the simplest models for rotating bodies. The physical properties of the general–relativistic global solution for the rigidly rotating disk of dust, which has been found recently as the solution of a boundary value problem, are discussed.

1 Introduction

Dust, as the simplest phenomenological material, is a good model for astrophysical and cosmological studies. As a source of gravitational fields, dust may be interpreted, in a hydrodynamical language, as a many–particle system, whose particles (mass–elements) interact via gravitational forces alone. While the cosmological relevance of that model has been demonstrated very early [1], investigations of isolated dust configurations are rather rare [2].

This paper is meant to discuss dust as a model for rotating bodies and to focus attention on the relativistic and ultrarelativistic behaviour of a rotating dust cloud. For this reason, we consider the relativistic generalization of the classical Maclaurin disk, which is the flattened (two–dimensional) limit of the famous Maclaurin spheroids. The corresponding global solution to the Einstein equations has recently been found as the solution of a boundary value problem [5], first formulated and approximately solved by Bardeen and Wagoner [3], [4]. In this connection, the appearance of a boundary value problem requires a comment: The reason for the complete absence of global solutions describing any uniformly rotating perfect fluid ball is that there is no systematic procedure for constructing solutions of the non–linear Einstein equations inside the source and matching them to exterior solutions

along an unknown ('free') surface such that those global solutions are regular everywhere. However, in the disk limit, in which the perfect fluid becomes dust[1], the body has no longer an interior region; it just consists of surface, so to say. Now, the surface conditions (vanishing pressure along the surface etc.) can be considered as the boundary conditions on a regular stationary and axisymmetric gravitational vacuum field (only the radius remains 'free') and this fact makes disk problems accessible to a systematic treatment. Namely, it has been shown that 'inverse' methods leading to linear structures (Riemann–Hilbert problems, linear integral equations) apply in this case. Following this idea, it was possible to describe the gravitational field of the rigidly rotating disk of dust mentioned above in terms of two integral equations [5]. One of them, the 'small' one, describes the behaviour of the gravitational field along the axis of symmetry and the physics on the disk (fields, mass–density, ...). It turned out that its solution can be represented in terms of elliptic functions [6]. The other one (the 'big' integral equation) makes use of the solution of the 'small' integral equation and describes the gravitational field everywhere. Surprisingly, this equation and its corresponding Riemann–Hilbert problem could be solved in terms of hyperelliptic functions [7].

In the next section we will briefly repeat the mathematical formulation of the boundary value problem and its solution. The main intent of this paper, however, is the discussion of the physical properties of the disk of dust solution in section 3. Finally, general conclusions form the last section.

2 The boundary value problem and its solution

To describe a rigidly rotating disk of dust we use cylindrical Weyl–Lewis–Papapetrou coordinates

$$ds^2 = e^{-2U}[e^{2k}(d\rho^2 + d\zeta^2) + \rho^2 d\varphi^2] - e^{2U}(dt + a d\varphi)^2, \tag{2.1}$$

which are adapted to stationary axisymmetric problems ($U = U(\rho, \zeta)$, $a = a(\rho, \zeta)$, $k = k(\rho, \zeta)$). In these coordinates, the vacuum Einstein equations are equivalent to the Ernst equation

$$(\Re f)(f,_{\rho\rho} + f,_{\zeta\zeta} + \frac{1}{\rho} f,_\rho) = f,_\rho^2 + f,_\zeta^2 \tag{2.2}$$

for the complex function

$$f(\rho, \zeta) = e^{2U} + ib \tag{2.3}$$

with

$$a,_\rho = \rho e^{-4U} b,_\zeta\,, \quad a,_\zeta = -\rho e^{-4U} b,_\rho \tag{2.4}$$

and

$$k,_\rho = \rho[U,_\rho^2 - U,_\zeta^2 + \frac{1}{4} e^{-4U}(b,_\rho^2 - b,_\zeta^2)], \quad k,_\zeta = 2\rho(U,_\rho U,_\zeta + \frac{1}{4} e^{-4U} b,_\rho b,_\zeta). \tag{2.5}$$

[1] Strictly speaking, the pressure – to – mass–density ratio tends to zero everywhere. (The maximum pressure – at the center – remains finite while the volume mass–density becomes infinite.) The radial distribution of the surface mass–density as well as the exterior gravitational field of this object become identical with those of a 'genuine' dust disk.

As a consequence of the Ernst equation (2.2), the integrability conditions $a,_{\rho\zeta} = a,_{\zeta\rho}$, $k,_{\rho\zeta} = k,_{\zeta\rho}$ are automatically satisfied and the metric functions a and k may be calculated from the Ernst potential f. Thus, it is sufficient to consider the Ernst equation alone.

The metric (2.1) allows an Abelian group of motions G_2 with the generators (Killing vectors)

$$\begin{aligned} \xi^i &= \delta^i_t, \quad \xi^i\xi_i < 0 \quad \text{(stationarity)}, \\ \eta^i &= \delta^i_\varphi, \quad \eta^i\eta_i > 0 \quad \text{(axisymmetry)}, \end{aligned} \tag{2.6}$$

where the Kronecker symbols δ^i_t and δ^i_φ indicate that ξ^i has only a t–component ($\xi^t = 1$) whereas η^i points into the azimuthal φ–direction (its trajectories have to form closed circles!). By applying (2.6) we get from (2.1) the invariant representations

$$e^{2U} = -\xi_i\xi^i, \quad a = -e^{-2U}\eta_i\xi^i \tag{2.7}$$

for the "Newtonian" gravitational potential U and the "gravitomagnetic" potential a.

To formulate the boundary value problem for an infinitesimally thin rigidly rotating disk of dust with a coordinate radius ρ_0 let us start from a spheroid–like rotating perfect fluid configuration (Fig. 1, left) and interpret the rigidly rotating disk of dust as an extremely flattened limiting case of that configuration (Fig. 1, right).

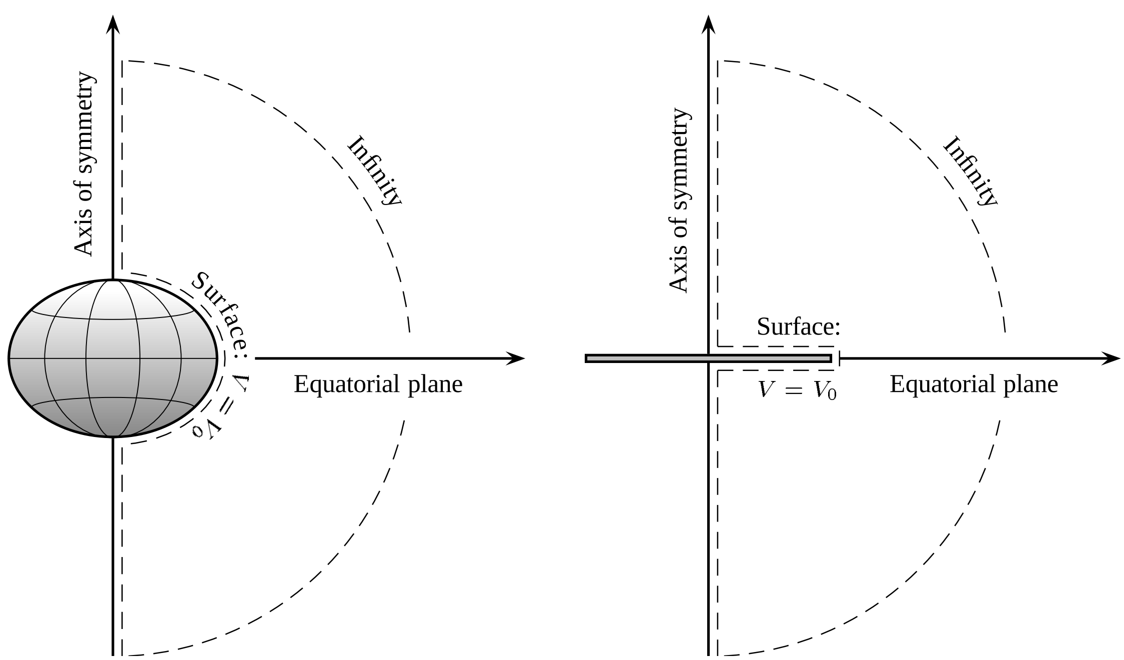

Fig. 1 — The rigidly rotating disk of dust (right–hand side) as the extremely flattened limit of a rotating perfect fluid body (left–hand side).

The hydrodynamics of a perfect fluid with the energy–momentum tensor

$$T_{ik} = (\epsilon + p)u_iu_k + pg_{ik}, \tag{2.8}$$

ϵ, p, u_i, g_{ik} being energy density, pressure, four–velocity, and metric, respectively, follows from the balance equation

$$T^{ik}{}_{;k} = 0, \tag{2.9}$$

where the semicolon denotes the covariant derivative. *Rotational* motion of the fluid means

$$u^i = e^{-V}(\xi^i + \Omega\eta^i), \quad u^i u_i = -1, \tag{2.10}$$

i.e., the four–velocity is a linear combination of the time–like Killing vector ξ^i and the azimuthal Killing vector η^i. Obviously,

$$e^{2V} = -(\xi^i + \Omega\eta^i)(\xi_i + \Omega\eta_i). \tag{2.11}$$

For *rigidly* rotating bodies, the angular velocity Ω is a constant,

$$\Omega = constant. \tag{2.12}$$

Then, for an equation of state

$$\epsilon = \epsilon(p), \tag{2.13}$$

the pressure p must be a function of V alone,

$$p = p(V). \tag{2.14}$$

This follows from Eqs. (2.8), (2.9). If Eq. (2.13) is surface–forming, the matching condition at the surface (fluid–vacuum interface) requires

$$p(V_0) = 0, \tag{2.15}$$

i.e. we have

$$V = V_0 \tag{2.16}$$

along the surface of the body. Eq. (2.11) inspires us to introduce a corotating frame of reference by the transformations

$$t' = t, \quad \varphi' = \varphi - \Omega t; \tag{2.17}$$

$$\xi^{i'} = \xi^i + \Omega\eta^i, \quad \eta^{i'} = \eta^i. \tag{2.18}$$

As a consequence of Eq. (2.18), the corotating potential f' constructed from the primed Killing vectors with the aid of the Eqs. (2.7) and (2.4) is again a solution of the Ernst equation,

$$(\Re f')(f'_{,\rho\rho} + f'_{,\zeta\zeta} + \frac{1}{\rho} f'_{,\rho}) = f'^{\,2}_{,\rho} + f'^{\,2}_{,\zeta}\,. \tag{2.19}$$

Hence, Eq. (2.16) tells us that the real part of the corotating Ernst potential is a constant along the surface of the body, $\Re f' = \exp(2V_0)$. We may assume the validity of this result for the disk limit, too,

$$\Re f'|_{\zeta=0^\pm} = e^{2V_0} \quad (0 \le \rho \le \rho_0), \tag{2.20}$$

where $\zeta = 0^\pm$ means top and bottom, respectively. Thus we have found our first boundary condition along the disk (above and below). As a consequence of the Einstein equations with the perfect fluid source (2.8) we may conclude in the disk limit that

$$(\rho^{-1} e^{4U'} a'_{,\rho})_{,\rho} + (\rho^{-1} e^{4U'} a'_{,\zeta})_{,\zeta} = 0 \tag{2.21}$$

holds everywhere *including* the disk. Applying the procedure known from the transition conditions in electrodynamics, one obtains

$$\left.\frac{\partial a'}{\partial \zeta}\right|_{\zeta=0^+} = \left.\frac{\partial a'}{\partial \zeta}\right|_{\zeta=0^-} \tag{2.22}$$

at all points of the surface. On the other hand, the reflectional symmetry of the gravitomagnetic potential a',

$$a'(\rho,\zeta) = a'(\rho,-\zeta) \tag{2.23}$$

leads to

$$\left.\frac{\partial a'}{\partial \zeta}\right|_{\zeta=0^+} = -\left.\frac{\partial a'}{\partial \zeta}\right|_{\zeta=0^-} \tag{2.24}$$

on the disk, so that $\partial a'/\partial\zeta$ has to vanish on the disk,

$$\left.\frac{\partial a'}{\partial \zeta}\right|_{\zeta=0^\pm} = 0 \quad (0 \le \rho \le \rho_0). \tag{2.25}$$

Combining this relation with Eq. (2.4), one has $b' = constant$ on the disk and, after a normalization,

$$\Im f'|_{\zeta=0^\pm} = b'|_{\zeta=0^\pm} = 0 \quad (0 \le \rho \le \rho_0) \tag{2.26}$$

as the other boundary condition along the disk (above and below). Since at infinity ($\rho^2+\zeta^2 \to \infty$) the space–time of any isolated source is Minkowskian, we have to take care of the relation

$$f|_{\rho^2+\zeta^2\to\infty} = 1. \tag{2.27}$$

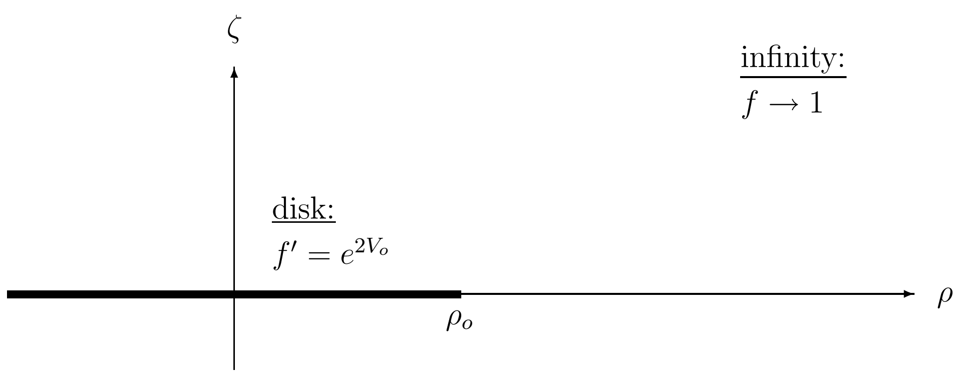

FIG. 2 — Boundary value problem [5]. f' is the Ernst potential in the corotating frame of reference defined by $\rho' = \rho$, $\zeta' = \zeta$, $\varphi' = \varphi - \Omega t$, $t' = t$ ($u^{i'} = e^{-V_o}\delta_4^{i'}$). The solution $f(\rho,\zeta)$ has to be regular everywhere outside the disk.

The conditions (2.20), (2.26) and (2.27) form our system of boundary conditions for an extremely flattened (infinitesimally thin) rigidly rotating disk consisting of a perfect fluid. It can be shown that the 2–dimensional mass elements of this disk move on the geodesics of their own field. This justifies the designation "disk of dust", cf. the footnote 1. Fig. 2 illustrates the boundary conditions on f. Moreover, the wanted Ernst potential satisfying those conditions has to be regular everywhere outside the disk. The axis of symmetry requires particular care, especially the elementary flatness condition along the axis ($f,_\rho(\zeta, \rho = 0) = 0$).

The boundary values depend on the two parameters $\exp(2V_0)$, cf. (2.20), and Ω, cf. (2.18), so that a 2–parameter solution can be expected from the very beginning. Ω and $\exp(2V_0)$ are "source" parameters with a clear physical meaning: Ω is the constant angular velocity of the mass elements (as measured by an observer at infinity), and $\exp(2V_0)$ defines the relative redshift

$$z_0 = e^{-V_0} - 1 \tag{2.28}$$

of a photon emitted from the center of the disk. Obviously, other parameter pairs can be chosen. It will turn out that the 'centrifugal' parameter

$$\mu = 2\Omega^2 \rho_0^2 \, e^{-2V_0} \tag{2.29}$$

is a good measure for the relativistic behaviour of the disk (μ varies between $\mu = 0$: Minkowski space and $\mu = \mu_0 = 4.62966\ldots$: ultrarelativistic case). We shall present the Ernst potential f in terms of μ and Ω or μ and ρ_0. On the other hand, the solution may also be characterized by the total mass M and the ζ–component of the angular momentum J, which far field quantities can be read off from a multipole expansion. Hence, a connection

$$\Omega = \Omega(M, J), \quad V_0 = V_0(M, J) \tag{2.30}$$

comparable with the parameter relations of "black hole thermodynamics" must hold.

Let us now briefly outline the solution procedure. We have made use of the so–called inverse scattering method of soliton physics, which was first utilized for the axisymmetric stationary vacuum Einstein equations by Maison [8], Belinski & Zakharov [9], Harrison [10], Neugebauer [11], Hauser & Ernst [12], Hoenselaers, Kinnersley & Xanthopoulos [13], and Aleksejew [14]. Some of these authors followed the line of Geroch [15], Kinnersley [16], Kinnersley & Chitre [17], and Herlt [18]. We have applied a local version [19] in which the Ernst equation (2.2) is the integrability condition of the "linear problem"

$$\Phi,_z = \left\{ \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} + \lambda \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \right\} \Phi, \tag{2.31}$$

$$\Phi,_{\bar{z}} = \left\{ \begin{pmatrix} \bar{D} & 0 \\ 0 & \bar{C} \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 0 & \bar{D} \\ \bar{C} & 0 \end{pmatrix} \right\} \Phi, \tag{2.32}$$

where $\Phi(z, \bar{z}, \lambda)$ is a 2×2 matrix depending on the spectral parameter

$$\lambda = \sqrt{\frac{K - i\bar{z}}{K + iz}} \quad (K \text{ a complex constant}) \tag{2.33}$$

as well as on the coordinates $z = \rho + i\zeta$, $\bar{z} = \rho - i\zeta$, whereas C, D and the complex conjugate quantities $\bar{C}$, $\bar{D}$ are functions of z, $\bar{z}$ (ρ, ζ) alone. Indeed, from $\Phi_{,z\bar{z}} = \Phi_{,\bar{z}z}$ and the formulae

$$\lambda_{,z} = \frac{\lambda}{4\rho}(\lambda^2 - 1), \quad \lambda_{,\bar{z}} = \frac{1}{4\rho\lambda}(\lambda^2 - 1) \tag{2.34}$$

it follows that a certain matrix polynomial in λ has to vanish. This yields the set of first order differential equations

$$C_{,\bar{z}} = C(\bar{D} - \bar{C}) - \frac{1}{4\rho}(C + \bar{D}), \quad D_{,\bar{z}} = D(\bar{C} - \bar{D}) - \frac{1}{4\rho}(D + \bar{C}) \tag{2.35}$$

plus the complex conjugate equations. The system (2.35) has the "first integrals"

$$D = \frac{f_{,z}}{f + \bar{f}}, \quad C = \frac{\bar{f}_z}{f + \bar{f}}. \tag{2.36}$$

Eliminating C and D in (2.35) one arrives at the Ernst equation (2.2). Vice versa, if f is a solution to the Ernst equation, the matrix Φ calculated from (2.31), (2.32) does not depend on the path of integration. The idea of the "inverse methods" is to discuss Φ, for fixed values of z, $\bar{z}$, as a holomorphic function of λ and to calculate C and D from Φ afterwards. This is an 'inverse' procedure compared with the 'normal' way which consists of the solution of differential equations with given coefficients. The term 'scattering' comes from the solution technique of the Korteweg–de Vries equation developed by Gardner, Greene, Kruskal & Miura [20] whose linear problem has partially the form of the (time–independent) Schrödinger equation. In this case, the calculation of the coefficients consists of the construction of the Schrödinger potential from the scattering data.

To construct Φ as a function of λ we have integrated the linear problem (2.31), (2.32) along the dashed line in Fig. 1 (right–hand side) and exploited the information of C, D and λ along the axis of symmetry ($f_{,\rho} = 0$, $\lambda = \pm 1$), the boundary conditions on the disk which simplify C' and D' (i.e. we had here to switch to the corotating system) and the simple structure of the linear problem at infinity ($C = D = 0$). In this way, we could pick up enough information to construct $\Phi(z, \bar{z}, \lambda)$ completely. (The crucial steps were the formulation and solution of a matrix Riemann–Hilbert problem in the complex λ–plane.) The linear problem (2.31), (2.32) tells us that Φ at $\lambda = 1$ may be normalized in a very simple way,

$$\Phi(z, \bar{z}, \lambda = 1) = \begin{pmatrix} \bar{f} & 1 \\ f & -1 \end{pmatrix}. \tag{2.37}$$

Hence, once Φ is known, the Ernst potential f can be read off from Φ ($f = \Phi_{21}(z, \bar{z}, 1)$).

The result for our problem is [7]

$$f = \exp\left\{\mu\left[\int\limits_{X_1}^{X_a} \frac{X^2\, dX}{W} + \int\limits_{X_2}^{X_b} \frac{X^2\, dX}{W} - \int\limits_{-i}^{i} \frac{hX^2\, dX}{W_1}\right]\right\}, \tag{2.38}$$

where the lower integration limits X_1, X_2 are given by

$$X_1^2 = \frac{i - \mu}{\mu}, \quad X_2^2 = -\frac{i + \mu}{\mu} \quad (\Re X_1 < 0, \quad \Re X_2 > 0). \tag{2.39}$$

whereas the upper limits X_a, X_b must be calculated from the integral equations

$$\int_{X_1}^{X_a} \frac{dX}{W} + \int_{X_2}^{X_b} \frac{dX}{W} = \int_{-i}^{i} \frac{hdX}{W_1}, \qquad \int_{X_1}^{X_a} \frac{X\,dX}{W} + \int_{X_2}^{X_b} \frac{X\,dX}{W} = \int_{-i}^{i} \frac{hX dX}{W_1}. \tag{2.40}$$

Here we have introduced the abbreviations

$$W = W_1 W_2, \qquad W_1 = \sqrt{(X - \zeta/\rho_0)^2 + (\rho/\rho_0)^2}, \qquad W_2 = \sqrt{1 + \mu^2(1 + X^2)^2} \tag{2.41}$$

and

$$h = \frac{\ln\left(\sqrt{1 + \mu^2(1 + X^2)^2} + \mu(1 + X^2)\right)}{\pi i \sqrt{1 + \mu^2(1 + X^2)^2}}\,. \tag{2.42}$$

The third integral in (2.38) as well as the integrals on the right-hand sides in (2.40) have to be taken along the imaginary axis in the complex X-plane with h and and W_1 fixed according to $\Re W_1 < 0$ (for ρ, ζ outside the disk) and $\Re h = 0$. The task of calculating the upper limits X_a, X_b in (2.39) from

$$u = \int_{-i}^{i} \frac{h\,dX}{W_1}\,, \quad v = \int_{-i}^{i} \frac{hX dX}{W_1} \tag{2.43}$$

is known as Jacobi's famous inversion problem. Göpel [21] and Rosenhain [22] were able to express the hyperelliptic functions $X_a(u, v)$ and $X_b(u, v)$ in terms of (hyperelliptic) theta functions. Later on it turned out that even the first two integrals in (2.38) can be expressed by theta functions in u and v! A detailed introduction into the related mathematical theory which was founded by Riemann and Weierstraß may be found in [23], [24], [25]. The representation of the Ernst potential (2.38) in terms of theta functions can be found in Stahl's book, see [23], page 311, Eq. (5). Here is the result: Defining a theta function $\vartheta(x, y; p, q, \alpha)$ by[2]

$$\vartheta(x, y; p, q, \alpha) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^{m+n} p^{m^2} q^{n^2} e^{2mx+2ny+4mn\alpha} \tag{2.44}$$

one can reformulate the expressions (2.38), (2.40) to give

$$f = \frac{\vartheta(\alpha_0 u + \alpha_1 v - C_1, \beta_0 u + \beta_1 v - C_2; p, q, \alpha)}{\vartheta(\alpha_0 u + \alpha_1 v + C_1, \beta_0 u + \beta_1 v + C_2; p, q, \alpha)}\; e^{-(\gamma_0 u + \gamma_1 v + \mu w)} \tag{2.45}$$

with u and v as in (2.43) and

$$w = \int_{-i}^{i} \frac{hX^2 dX}{W_1}\,. \tag{2.46}$$

The normalization parameters α_0, α_1; β_0, β_1; γ_0, γ_1, the moduli p, q, α of the theta function and the quantities C_1, C_2 are defined on the two sheets of the hyperelliptic Riemann surface related to

$$W = \mu\sqrt{(X - X_1)(X - \bar{X}_1)(X - X_2)(X - \bar{X}_2)(X - i\bar{z}/\rho_0)(X + iz/\rho_0)}, \tag{2.47}$$

see Figure 3.

[2] We use $\vartheta = \vartheta \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ where the bracket indicates the characteristic, see [24].

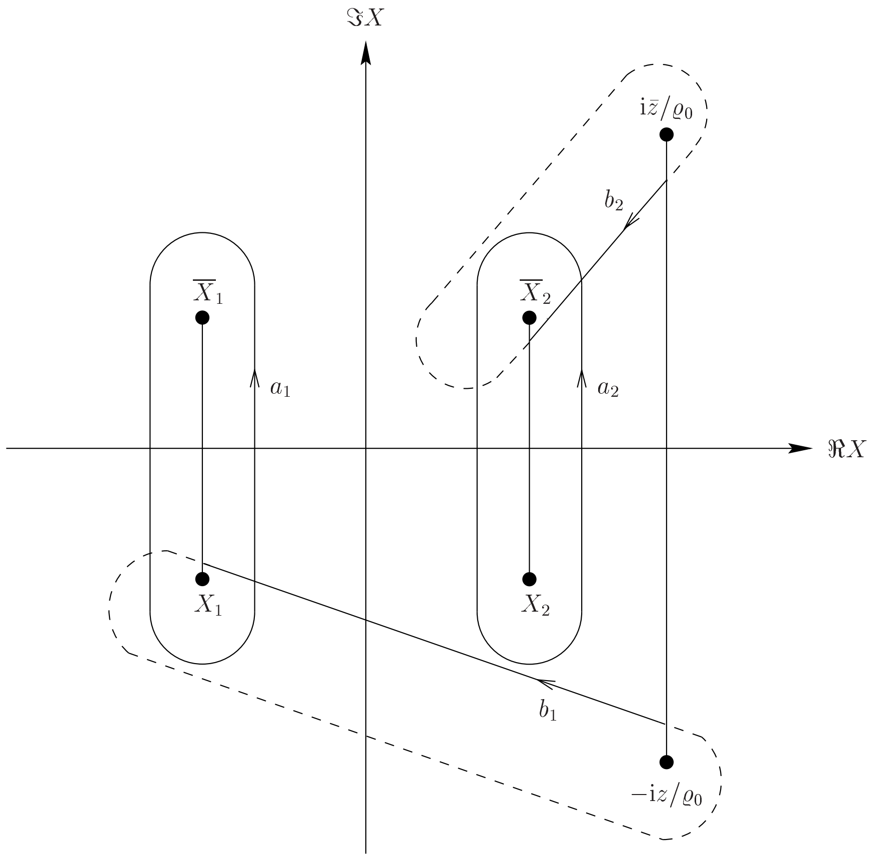

FIG. 3 — Riemann surface with cuts between the branch points X_1 and $\bar{X}_1$, X_2 and $\bar{X}_2$, $-iz/\rho_0$ and $i\bar{z}/\rho_0$. Also shown are the four periods a_i and b_i $(i = 1, 2)$. (Continuous/dashed lines belong to the upper/lower sheet defined by $W \to \pm\mu X^3$ as $X \to \infty$.)

There are two normalized Abelian differentials of the first kind

$$d\omega_1 = \alpha_0 \frac{dX}{W} + \alpha_1 \frac{X dX}{W} \tag{2.48}$$

$$d\omega_2 = \beta_0 \frac{dX}{W} + \beta_1 \frac{X dX}{W} \tag{2.49}$$

defined by

$$\oint_{a_m} d\omega_n = \pi i \, \delta_{mn} \quad (m = 1, 2;\, n = 1, 2)\,. \tag{2.50}$$

Eq. (2.50) consists of four linear algebraic equations and yields the four parameters α_0, α_1, β_0, β_1 in terms of integrals extending over the closed (deformable) curves a_1, a_2. It can be shown that there is one normalized Abelian differential of the third kind

$$d\omega = \gamma_0 \frac{dX}{W} + \gamma_1 \frac{X dX}{W} + \mu \frac{X^2 dX}{W} \tag{2.51}$$

with vanishing $a-$periods,

$$\oint_{a_j} d\omega = 0 \quad (j = 1, 2)\,. \tag{2.52}$$

This equation defines γ_0, γ_1 (again via a linear algebraic system). The Riemann matrix

$$(B_{ij}) = \begin{pmatrix} \ln p & 2\alpha \\ 2\alpha & \ln q \end{pmatrix} \quad (i = 1, 2;\ j = 1, 2) \tag{2.53}$$

(with negative definite real part) is given by

$$B_{ij} = \oint_{b_i} d\omega_j \tag{2.54}$$

and defines the moduli p, q, α of the theta function (2.44). Finally, the quantities C_1, C_2 can be calculated by

$$C_i = -\int_{-iz/\rho_0}^{\infty^+} d\omega_i \quad (i = 1, 2)\,, \tag{2.55}$$

where + denotes the upper sheet. Obviously, all the quantities entering the theta functions and the exponential function in (2.45) can be expressed in terms of well–defined integrals and depend on the three parameters ρ/ρ_0, ζ/ρ_0, μ. The corresponding "tables" for α_i, β_i, γ_i, C_i, B_{ij}, u, v, w can easily be calculated by numerical integrations. Fortunately, theta series like (2.44) converge rapidly. For $0 < \mu < \mu_0$, the solution (2.45) is analytic *everywhere* outside the disk – even at the rings $-iz/\rho_0 = X_1$, X_2.

Figures 4 and 5 give an impression of the Ernst potential for $\mu = 3$, as an example.

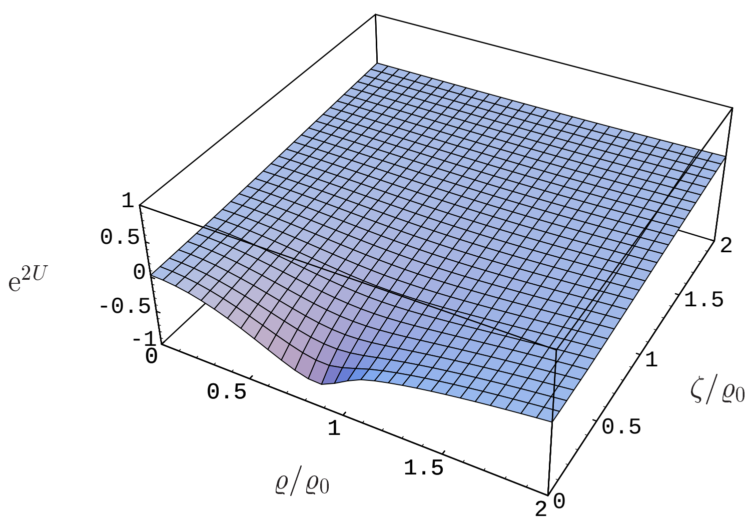

FIG. 4 — The real part (e^{2U}) of the Ernst potential for $\mu = 3$.

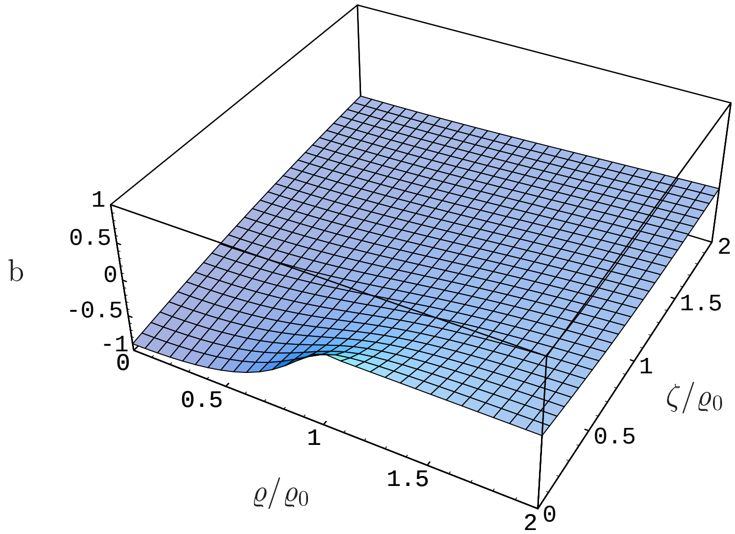

FIG. 5 — The imaginary part (b) of the Ernst potential for $\mu = 3$.

3 Physical properties

In dimensionless coordinates, ρ/ρ_0, ζ/ρ_0, the solution depends on the single parameter μ. The original parameter V_0 entering our boundary value problem can be calculated from μ via $\Re f(\rho = 0, \zeta = 0^+)$ (on the axis $\rho = 0$ we have $\Re f' = \Re f$) leading to [6]

$$V_0 = -\frac{1}{2}\sinh^{-1}\left\{\mu + \frac{1+\mu^2}{\wp[I(\mu); \frac{4}{3}\mu^2 - 4, \frac{8}{3}\mu(1+\frac{\mu^2}{9})] - \frac{2}{3}\mu}\right\}, \tag{3.1}$$

$$I(\mu) = \frac{1}{\pi}\int_0^\mu \frac{\ln(x + \sqrt{1+x^2})dx}{\sqrt{(1+x^2)(\mu - x)}}, \tag{3.2}$$

where $\wp$ is the Weierstraß function defined by

$$\int_{\wp(x;g_2,g_3)}^\infty \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}} = x. \tag{3.3}$$

Fig. 6 shows $\exp[2V_0(\mu)]$ in the range $0 < \mu < \mu_0 = 4.62966184\ldots$ with μ_0 being the first zero of the denominator in (3.1). This corresponds to $0 > V_0 > -\infty$, where $|V_0| \ll 1$ is the Newtonian limit. Note that, according to (2.29) and (3.1), $\Omega\rho_0$ is a given function of μ, and we can use either ρ_0 or Ω as the second parameter.

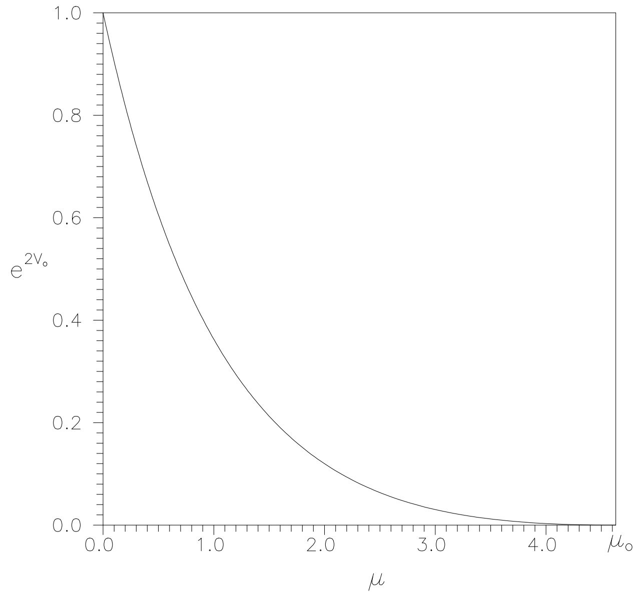

FIG. 6 — The corotating Ernst potential $f' = e^{2V_0}$ in the disk as a function of the parameter μ.

In Fig. 7 the invariant baryonic (proper) surface mass–density σ_p can be found as a function of ρ/ρ_0 for several values of μ. Note that the volume mass–density ϵ entering the energy–momentum tensor ($T_{ik} = \epsilon u_i u_k$) may be expressed formally by

$$\epsilon = \sigma_p(\rho) e^{U-k} \delta(\zeta), \tag{3.4}$$

where $\delta(\zeta)$ is the usual Dirac delta–distribution.

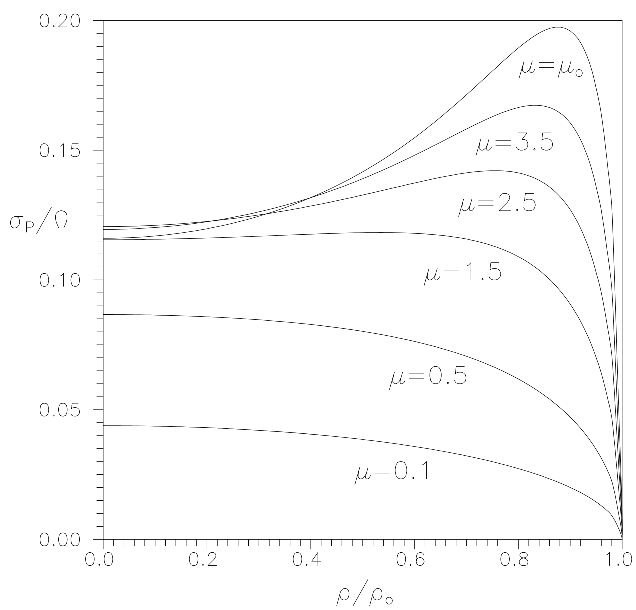

FIG. 7 — The surface mass–density σ_p. The normalized quantity σ_p/Ω is shown as a function of ρ/ρ_0.

σ_p can be calculated from

$$\sigma_p = \frac{1}{2\pi}\, e^{U-k}\, \frac{\partial U'}{\partial \zeta}\bigg|_{\zeta=0^+} . \tag{3.5}$$

A more explicit expression has been given in [6]. From σ_p we can calculate the total baryonic mass M_0, the gravitational mass M, and the total angular momentum J:

$$M_0 = \int\limits_\Sigma \epsilon\sqrt{-g}\, u^4 d^3x = 2\pi\, e^{-V_0} \int\limits_0^{\rho_0} \sigma_p e^{k-U} \rho d\rho, \tag{3.6}$$

$$M = 2\int\limits_\Sigma (T_{ab} - \frac{1}{2}T g_{ab}) n^a \xi^b dV = 2\pi \int\limits_0^{\rho_0} \sigma_p e^{k-U} \rho d\rho + 4\pi\Omega e^{-V_0} \int\limits_0^{\rho_0} \sigma_p e^{k-U} u^i \eta_i \rho\, d\rho, \tag{3.7}$$

$$J = -\int\limits_\Sigma T_{ab} n^a \eta^b dV = 2\pi\, e^{-V_0} \int\limits_0^{\rho_0} \sigma_p e^{k-U} u^i \eta_i \rho\, d\rho, \tag{3.8}$$

where Σ is the spacelike hypersurface $t = constant$ with the unit future–pointing normal vector n^a. Note that $M = \exp(V_0)M_0 + 2\Omega J$, cf. [26]. Alternatively, M and J – as the first gravitational multipole moments – may be obtained from the asymptotic expansion of the Ernst potential, e.g. on the symmetry axis. The dependence of M_0, M and J on μ can be seen in Fig. 8.

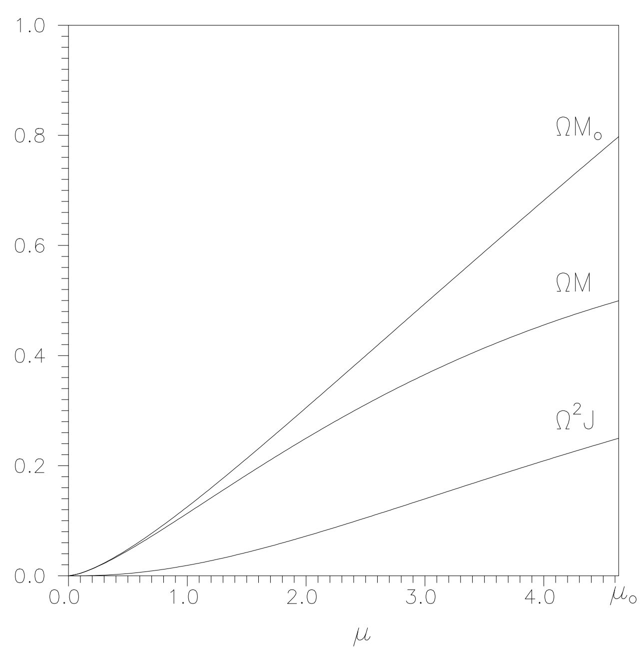

FIG. 8 — Baryonic mass M_0, gravitational mass M and angular momentum J. The normalized quantities ΩM_0, ΩM and $\Omega^2 J$ are shown in dependence on μ.

The relative binding energy $(M_0 - M)/M_0$ as well as the characteristic quantity M^2/J are shown in Fig. 9.

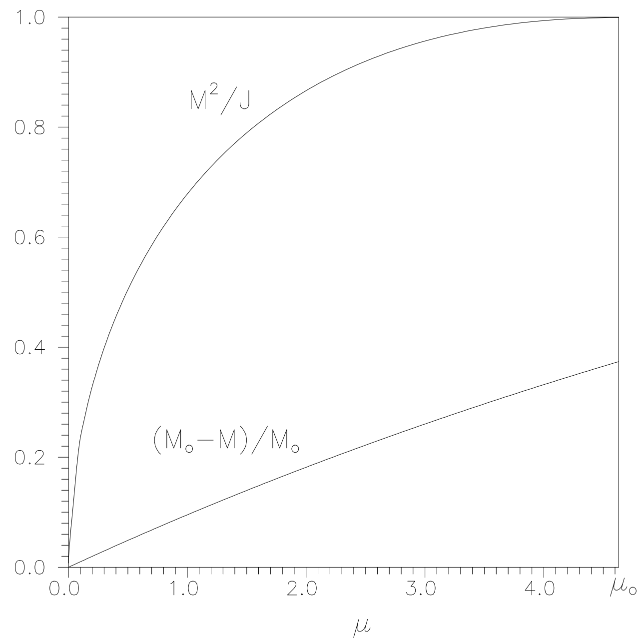

FIG. 9 — Relative binding energy and M^2/J.

For $\mu \to \mu_0$, M and J together with all the other multipole moments approach exactly the values of the extreme Kerr solution ($\Omega = 1/\,2M$ may be identified with the angular velocity of the horizon) [30]. In fact, the solution becomes identical with the extreme Kerr solution for all values of ρ and ζ except $\rho = \zeta = 0$, which represents the horizon of the extreme Kerr black hole. Note that, for non–vanishing Ω (finite M), $\rho_0 \to 0$ as $\mu \to \mu_0$.

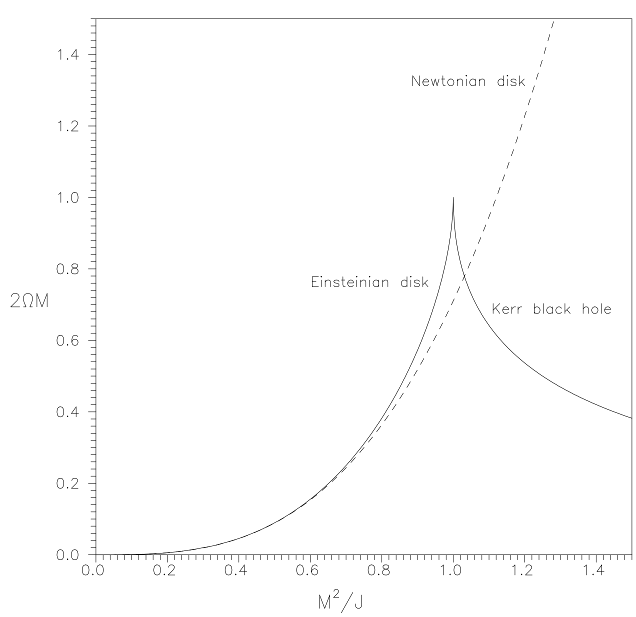

FIG. 10 — Relation between ΩM and M^2/J for the classical Maclaurin disk (dashed line), the general–relativistic dust disk and the Kerr black–hole [5].

Fig. 10 combines the parameter relations between ΩM and M^2/J for the dust disk as well as for the Kerr black–hole. Both branches are connected at the point $M^2/J = 1$.

Another limit of the space–time for $\mu \to \mu_0$ is obtained for finite values of ρ/ρ_0 and ζ/ρ_0, see also [4]. The interpretation of the solution for $\mu > \mu_0$ is beyond the scope of this paper. We only want to mention here, that there are further zeros μ_n $(n = 1, 2, \dots)$ of the denominator in (3.1) leading always to the extreme Kerr metric ($\mu_1 = 38.70908\dots$, $\mu_2 = 176.92845\dots$, $\dots$).

A characteristic feature of relativistically rotating bodies are dragging effects due to the gravitomagnetic potential. Dragging effects near the rigidly rotating disk of dust have been discussed in [31]. In particular, the dust disk generates an ergoregion for $\mu > \mu_e = 1.68849\dots$, see Fig. 11. In this region the Killing vector ξ^i becomes spacelike. As a consequence, $d\varphi/dt > 0$ must hold for any timelike worldline there. Thus, seen from infinity, any observer inside the ergoregion is forced to rotate in the same direction as the disk. For $\mu \to \mu_0$ the well–known ergosphere of the extreme Kerr black hole appears.

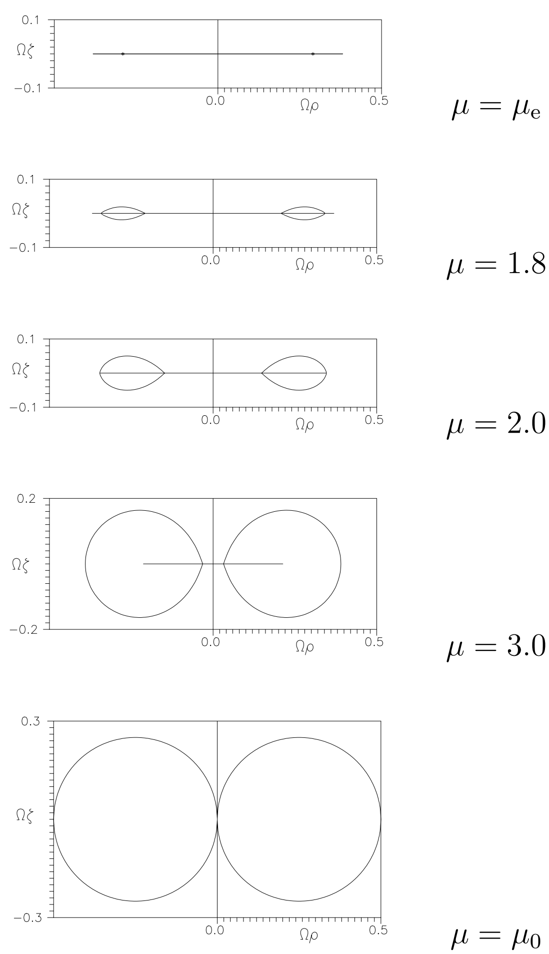

FIG. 11 — The ergoregion [31].

Note that the shape of the ergoregion for $\mu = 3$ may be rediscovered in Fig. 4.

4 Conclusions

In this article, we have considered the rigidly rotating disk of dust as an extremely flattened rotating perfect fluid body. Such controllable limiting procedures reducing the dimension of a body (here the thickness) are important for the derivation of equations of motion of particles from the dynamics of extended bodies. In our case, the motion of the two–dimensional mass elements is generally geodesic and independent of the underlying perfect fluid model. In that sense, our disk of dust, like the classical Maclaurin disk, represents a “universal” limit for any rigidly rotating perfect fluid ball. There is faint hope of an explicit global solution for three–dimensional rotating perfect fluid sources with the limit (2.45)!

Another aspect of our solution is its derivation from a boundary value problem by means of the inverse scattering method. It may be expected that this method will prove to be a powerful tool for the solution of other boundary value problems for axisymmetric stationary gravitational vacuum fields in Einstein's theory. It could also improve the insight into the structure of the exterior fields of rotating bodies. A first step in this direction consisted in the treatment of the reflectional symmetry of the gravitational field with the aid of the linear problem (2.31), (2.32) in [27].

Finally, as a ‘practical’ application, the solution could be used as a testbed for numerical codes describing rotating star models in general relativity, as, e.g., neutron stars.

There is also a formal aspect to be mentioned. Generalizing the expressions (2.38), (2.40) it was possible to construct a solution class in terms of hyperelliptic theta functions containing an arbitrary potential function and – depending on the genus – an arbitrary number of constants [28]. The relation of this class to the finite–gap class of solutions [29] requires a subtle discussion of certain limiting procedures. We want to emphasize that it is not possible to know *a priori* into which class of (known or unknown) solutions of the Ernst equation a boundary value problem might fall.

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